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Dualities for Torsion-Free Abelian Groups of Finite Rank

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Two well-known dualities have been very useful in the study of torsion-free abelian groups of finite rank: Warfield duality for locally free groups and Arnold duality for quotient divisible groups [Wa, Ar]. In this note we establish a duality, on classes of torsion-free abelian groups of finite rank, which generalizes both Warfield and Arnold duality. Our results were inspired by a recent paper of Fomin [Fo2], who constructed a duality which is also a special case of the one presented here.

The basic idea is to write a torsion-free abelian group of finite rank (hereafter, "group") G as a sum $G = G_1 + G_2$, where G_1 is locally free and G_2 is quotient divisible. A dual for G is then obtained by adding the Warfield dual of G_1, $W(G_1)$, and the Arnold dual of G_2, $A(G_2)$ (inside of $\mathrm{Hom}(G, Q)$). The idea of breaking up G into a locally free and a quotient divisible part is not a new one—it was investigated by Murley in [Mu]. As noted by Murley, one easy way to obtain G_1 and G_2 is to take a full free subgroup F of G and write $G/F = D \oplus R$, where D is a divisible and R is a reduced torsion group. Then choose subgroups G_1 and G_2 of G containing F so that $G_1/F = R$ and $G_2/F = D$. In general, the G_1 obtained in this way is decidedly non-unique, even up to quasi-isomorphism. It is therefore somewhat surprising that we can form $G^* = W(G_1) + A(G_2)$ to obtain a group which, up to quasi-equality, is independent of the choice of G_1.

Further complexity can be introduced due to the fact that the Warfield and Arnold duals treat free and divisible localizations differently, prompting our definition of PX-groups (Definition 2.1). On the category of PX-groups and quasi-homomorphisms (in which objects are quasi-equality classes), $G \to G^*$ defines an exact contravariant functor (Theorem 2.13). The PX-group G^* has a special property, which was also studied by Murley. On the PX-groups with this special property, the "dualizable PX-groups," $*$ determines a duality (Theorem 3.3), which generalizes the dualities of Warfield, Arnold, and Fomin (Proposition 3.5). We remark

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that our results hold also in the setting of modules over Dedekind domains, since both Arnold and Warfield duality can still be applied in this context [La1, La2, Re]. However, to keep the arguments as simple as possible, we have clung to the more familiar skirts of abelian groups.

Throughout, G will denote a (torsion-free finite rank abelian) group, and F will denote a full free subgroup of G. If S is a subset of the rational primes, and A is any abelian group, then $A_S \simeq Z_S \otimes A$ is the usual localization of A at S. If A is torsion-free, we regard $A \subset A_S$. The symbols $\dot{\cong}$, $\dot{=}$, and $\dot{\subset}$ denote quasi-isomorphism, quasi-equality, and quasi-containment, respectively. A subgroup H of G is called *full* in G provided G/H is torsion.

If A/Z is a subgroup of Q/Z, the type of A/Z is the type of A. The *Richman type of a group* G, $RT(G)$, is the quasi-isomorphism class of the torsion group G/F. The *outer type of* G, $OT(G)$, is type(X), where X is a subgroup of Q containing Z such that $(X/Z)_p$ is the maximal cocyclic summand appearing in the p-component of G/F (see [Wa]). Define the *finite outer type of* G, $FOT(G)$, to be the outer type of $(G/F)/\mathrm{div}(G/F)$, the reduced part of G/F. That is, $FOT(G) = \mathrm{type}(X)$ where X is a subgroup of Q containing Z such that $(X/Z)_p$ is the maximal cyclic summand appearing in the p-component of the reduced part of G/F. Since $RT(G)$ is a quasi-isomorphism invariant of the group G (see [Ri1]), so are $OT(G)$ and $FOT(G)$. The group G is *quotient divisible*, as defined in [BP], provided G/F is quasi-equal to a divisible group, equivalently, $FOT(G) = \mathrm{type}(Z)$.

1. Arnold and Warfield Duality

For the reader's convenience, we review the dualities of Warfield and Arnold. Let X be a subgroup of Q. A torsion-free abelian group G of finite rank is called *X-locally free* provided $OT(G) \leqslant \mathrm{type}(X)$ and $pX = X$ if and only if $pG = G$. Warfield [Wa] showed that the functor Hom(—, X) defines an exact duality on the category of X-locally free groups and homomorphisms. In particular, $G \cong \mathrm{Hom}(\mathrm{Hom}(G, X), X)$ under the natural (evaluation) map. We will make frequent use of the fact that if G is X-locally free and S is any set of rational primes, then $\mathrm{Hom}(G, X)_S = \mathrm{Hom}(G_S, X_S)$.

To define Arnold duality, for each prime p we identify a group G with its image $1 \otimes G$ in $\hat{Q}_p \otimes G$ (all tensor products are over Z). If G is a quotient divisible group then G is defined, up to quasi-equality, by the divisible hull QG and the collection of local invariants, $L_p(G) = \mathrm{div}(\hat{Z}_p \otimes G)$ over all primes p. Specifically, let F be a full free subgroup of a quotient divisible group G. Then $G \doteq \bigcap G_p$, where $G_p = QG \cap ((\hat{Z}_p \otimes F) + \mathrm{div}(\hat{Z}_p \otimes G))$. (See [BP] or [La1]).

The Arnold dual $A(G)$ of a quotient divisible group G is defined by specifying the divisible hull $QA(G) = \mathrm{Hom}(G, Q) = \mathrm{Hom}(QG, Q)$ and the local invariants $\mathrm{div}(\hat{Z}_p \otimes A(G)) = L_p(A(G))$ as follows. Let $\theta\colon \mathrm{Hom}(\hat{Q}_p \otimes G, \hat{Q}_p) \to \hat{Q}_p \otimes \mathrm{Hom}(G, Q)$ be the natural isomorphism, and set $L_p(A(G)) = \theta(\mathrm{div}(\hat{Z}_p \otimes G)^{\perp}) = \theta(\{f\colon \hat{Q}_p \otimes G \to \hat{Q}_p \mid f(\mathrm{div}(\hat{Z}_p \otimes G)) = 0\})$. Then $A(G)$ is a well defined (up to quasi-equality) quotient divisible subgroup of $\mathrm{Hom}(G, Q)$ and $G \doteq A(A(G))$ under the natural embedding $G \to \mathrm{Hom}(\mathrm{Hom}(G, Q), Q)$. The fact that p-rank $A(G) = \text{rank } G - p\text{-rank } G$ will be used repeatedly. Also, it will be convenient to omit mention of the isomorphism θ and simply identify $\mathrm{Hom}(\hat{Q}_p \otimes G, \hat{Q}_p)$ with $\hat{Q}_p \otimes \mathrm{Hom}(G, Q)$. For more details, see [La1].

2. PX-Groups and the Functor $*$

As mentioned above, some additional complexity can be introduced by the fact that the Warfield and Arnold duals treat free and divisible localizations differently. Specifically, if S is a set of primes such that G_S is a free Z_S-module, then $W(G_S)$ will be a free Z_S-module, while $A(G_S)$ will be S-divisible. Similarly, if G_S is divisible, then $W(G_S)$ is divisible, while $A(G_S)$ is a free Z_S-module. Roughly speaking, such a set of primes S can be "assigned" to either the locally free or the quotient divisible part of G. This is the motivation for our next definition, the definition of PX-groups. The definition is slightly technical, but the conditions are precisely those we need. We will employ the following notation:

Π = set of integral primes,

$I = I(G) = \{p \mid (G/F)_p \text{ is infinite}\}$ (the "infinite" primes),

$D = D(G) = \{p \mid pG = G\}$ (the "divisible" primes),

$\Delta = \Delta(G) = I \setminus D$ (the "difference").

Definition 2.1. Let $P \subset \Pi$ and let X be a subgroup of Q. A group G is called a *PX-group* provided

(a) $FOT(G) \leqslant \mathrm{type}(X)$,

(b) $pX = X$ if and only if $p \in D(G) \setminus P$,

(c) $\Delta = \Delta(G) \subset P$.

Lemma 2.2. (a) *Every group G is a PX-group for some P and X.*

(b) *The class of PX-groups is closed under quasi-isomorphism, sums, summands, and homomorphic images.*

Proof. (a) Let $P = I(G)$ and $\mathrm{type}(X) = FOT(G)$. (b) Routine.

EXAMPLE 2.3. Let G be a rank-2 group with $F = Z \oplus Z \subset A \oplus Z \subset G \subset Q \oplus Q$ where $A = \langle \{1/p \mid p \text{ a prime}\} \rangle$, $A \oplus 0$ and $0 \oplus Z$ are pure in G, G is strongly indecomposable, and $G/F = \bigoplus_{p \in \Pi} Z(p^\infty)$. Such a group is not difficult to construct. Then there is an exact sequence $0 \to A \to G \to Q \to 0$. Now G is a PX-group if $P = \Pi$ and $X = Z$, while the pure subgroup A is not a PX-group since $FOT(A) \leqslant \text{type}(X)$ fails.

If G is a PX-group and F is a full free subgroup of G, write $G/F = R \oplus \bar{D}$ where $\bar{D}$ is divisible and R is reduced. Define full subgroups G_1 and G_2 of QG, containing F, by

$$G_2/F = \bar{D}_P, \qquad G_1/F = R \oplus \bar{D}_{\Pi \setminus P}. \tag{2.4}$$

Clearly, $G = G_1 + G_2$. Note that the subgroup G_2 is uniquely defined given F, and therefore is unique up to quasi-equality inside G. The definition of the subgroup G_1 depends both on the choice of F and the choice of the reduced summand R. Hence, in general, G_1 is not uniquely defined up to quasi-equality. However, Definition 2.4 implies $(G_1/F)_{\Pi \setminus \Delta} = R_{\Pi \setminus \Delta} \oplus \bar{D}_{\Pi \setminus P} = R_{P \setminus \Delta} \oplus R_{\Pi \setminus P} \oplus \bar{D}_{\Pi \setminus P} = R_{P \setminus I} \oplus (G/F)_{\Pi \setminus P} = (G/F)_{P \setminus I} \oplus (G/F)_{\Pi \setminus P}$. Thus, $(G_1)_{\Pi \setminus \Delta}$ depends only on the choice of F, and so, like G_2, is unique up to quasi-equality.

If G is a PX-group and G_1, G_2 are subgroups satisfying (2.4), then $G = G_1 + G_2$ is called a *PX-decomposition* of G.

EXAMPLE 2.5. Let G be a subgroup of Q with $Z \subset G$. In this case, $I = D = \{p \mid pG = G\}$, whence $\Delta = \phi$. Suppose P is any subset of the primes. Denote by G_1 the subgroup of G such that $G_1/Z = (G/Z)_{\Pi \setminus (I \cap P)}$ and by G_2 the subgroup of G such that $G_2/Z = (G/Z)_{I \cap P}$. Then if $Z \subset X$ is a subgroup of Q, G is a PX-group if and only if $G_1 \doteq X$ and $\{p \mid pX = X\} = I \setminus P$. In this case, $G = G_1 + G_2$ is a PX-decomposition.

Next suppose that G is a PX-group which is a Butler group, that is, an epimorphic image of a finite direct sum of subgroups of Q. Let H be a pure rank one subgroup of G. We claim H is a PX-group. The only non-trivial part of the claim is that $FOT(H) \leqslant \text{type}(X)$. We prove $FOT(H) \leqslant FOT(G) \leqslant \text{type}(X)$ by induction on rank G for rank $G > 1$. Suppose, by way of contradiction, that $FOT(H) \not\leqslant FOT(G)$. Clearly, $FOT(H) \leqslant OT(H) \leqslant OT(G)$. Thus, there must exist an infinite set of primes S such that $FOT(H_S) = OT(H_S) > FOT(G_S)$ and $OT(G_S) = \text{type}(Q)$. Suppose $A_1 \oplus \cdots \oplus A_n \to G$ is an epimorphism with $A_i \subset Q$. Without loss of generality, we may assume $0 \neq \text{Image } A_i$ is pure in G. In this case, there must be an index i and an infinite subset T of S such that $FOT(H_T) > FOT(G_T)$ and $(A_i)_T = Q$. Therefore, $\text{div}(G_T) \neq 0$, and we may write $G_T = \text{div}(G_T) \oplus K$, where $H_T \subset K$. However, H_T is then a pure rank one subgroup of the Butler group K so that $FOT(H_T) \leqslant FOT(K)$ by the induction hypothesis. Since $FOT(K) \leqslant FOT(G_T)$, we have arrived at a contradiction. Thus, $FOT(H) \leqslant FOT(G)$, completing the proof of the claim. In particular, each of the A_i's above is a PX-group. Now if $A_i = A_{i1} + A_{i2}$ is the PX-decomposition of the rank one PX-group A_i, then $G_1 = \mathrm{Im}(A_{11} \oplus \cdots \oplus A_{n1})$ and $G_2 = \mathrm{Im}(A_{12} \oplus \cdots \oplus A_{n2})$ give a PX-decomposition, $G = G_1 + G_2$, of G. Note that both G_1 and G_2 are Butler groups.

Remark. If H is any pure subgroup of a Butler PX-group G, then H is a finite sum of rank one pure subgroups, each of which is a PX-group by the arguments in Example 2.5. Consequently, H is a homomorphic image of a PX-group, and therefore a PX-group by Lemma 2.2(b). Thus, in contrast to Example 2.3, pure subgroups of Butler PX-groups are PX-groups.

PROPOSITION 2.6. *Let G be a PX-group and $G = G_1 + G_2$ a PX-decomposition of G. Then G_1 is X-locally free and G_2 is quotient divisible. If $G^* = \mathrm{Hom}(G_1, X) + A((G_2)_P)$ in $\mathrm{Hom}(G, Q)$, then G^* is independent of the choice of G_1, and hence, up to quasi-equality is a uniquely defined subgroup of $\mathrm{Hom}(G, Q)$.*

Proof. The subgroup G_2 is, by definition, quotient divisible. Moreover, conditions (a)–(c) of the definition of PX-group guarantee that G_1 is X-locally free. Let F be the fixed full free subgroup of G with respect to which G_1 and G_2 are defined and regard $\mathrm{Hom}(F, Z)$ as a fixed full free subgroup of G^*. Since $A((G_2)_P)$ is uniquely defined up to quasi-equality, we restrict our attention to the group $\mathrm{Hom}(G_1, X)$. As observed after (2.4), if p is a prime not belonging to $\Delta = \Delta(G)$, then the localization $(G_1)_p$ is uniquely defined, and therefore so is the localization $\mathrm{Hom}(G_1, X)_p = \mathrm{Hom}((G_1)_p, X_p)$. Thus, we need concern ourselves only with a fixed prime $p \in \Delta$. For any group A, let $\hat{A} = \hat{Z}_p \otimes A$. Note that $\hat{G}_2 = L_p + \hat{F}$, where $L_p = \mathrm{div}(\hat{G}_2)$. Hence, there is an exact sequence

$$\text{(i)} \quad 0 \to L_p \to \hat{G}_2 \to \hat{F}/(\hat{F} \cap L_p) \to 0.$$

Denote $E_p = \hat{F}/(\hat{F} \cap L_p)$. Then there is a split exact sequence

$$\text{(ii)} \quad 0 \to \hat{F} \cap L_p \to \hat{F} \to E_p \to 0$$

which induces

$$\text{(iii)} \quad 0 \to \mathrm{Hom}(E_p, \hat{Z}_p) \to \mathrm{Hom}(\hat{F}, \hat{Z}_p) \to \mathrm{Hom}(\hat{F} \cap L_p, \hat{Z}_p) \to 0.$$

Next, using (i), we identify $\mathrm{Hom}(E_p, \hat{Q}_p)$ with $(L_p)^{\perp} = \mathrm{div}(\hat{Z}_p \otimes A((G_2)_P))$ and use (iii) to produce a split exact sequence

$$\text{(iv)} \quad 0 \to \mathrm{Hom}(E_p, \hat{Q}_p) \to \hat{Z}_p \otimes A((G_2)_P) \to \mathrm{Hom}(\hat{F} \cap L_p, \hat{Z}_p) \to 0$$

since $\hat{Z}_p \otimes A((G_2)_P) = \mathrm{Hom}(E_p, \hat{Q}_p) + \mathrm{Hom}(\hat{F}, \hat{Z}_p)$.

Now $G_1 \cap G_2 = F$, so $\hat{G}_1 \cap \hat{G}_2 = \hat{F}$ and $\hat{F} \cap L_p$ is pure in $\hat{F}$, and hence in $\hat{G}_1$, since $Q(\hat{F} \cap L_p) \subset \hat{G}_2$. Thus, there is an exact sequence

$$\text{(v)} \quad 0 \to \hat{F} \cap L_p \to \hat{G}_1 \to K \to 0,$$

where the quotient K may be identified with a submodule of QE_p (see (ii)). Denote $W(G_1) = \operatorname{Hom}(G_1, X)$. Since $p \in \Delta$, $(G_1)_p$ and X_p are free Z_p-modules and we may identify $\hat{Z}_p \otimes W(G_1)$ with $\operatorname{Hom}(\hat{G}_1, \hat{X})$, where $\hat{X} = \hat{Z}_p X \subset \hat{Q}_p$. Thus, applying $\operatorname{Hom}(\text{—}, \hat{X})$ to (v) yields

$$\text{(vi)} \quad 0 \to \operatorname{Hom}(K, \hat{X}) \to \hat{Z}_p \otimes W(G_1) \to \operatorname{Hom}(\hat{F} \cap L_p, \hat{X}) \to 0.$$

Finally, we may combine (iv) and (vi) to obtain the split exact sequence

$$\text{(vii)} \quad 0 \to \operatorname{Hom}(E_p, \hat{Q}_p) \to \hat{Z}_p \otimes [W(G_1) + A((G_2)_P)] \to \operatorname{Hom}(\hat{F} \cap L_p, \hat{X}) \to 0.$$

This final sequence shows that $(G^*)_p = [W(G_1) + A((G_2)_P)]_p = (\hat{Z}_p \otimes G^*) \cap QG^* = [\operatorname{Hom}(E_p, \hat{Q}_p) + \operatorname{Hom}(\hat{F}, \hat{X})] \cap \operatorname{Hom}(G, Q)$ is independent of the choice of G_1, and the proposition follows.

The exact sequence (vii) of the proof in fact carries some additional information about G^*. It shows that if $p \in \Delta$, then $\operatorname{Hom}(F_p, X_p) \subset (G^*)_p$ and that $(G^*)_p/\operatorname{Hom}(F_p, X_p)$ is divisible. It follows that $(G^*)_\Delta$ contains a full homogeneous completely decomposable subgroup, $\operatorname{Hom}(F_\Delta, X_\Delta)$, of type equal to type$(X_\Delta)$. Moreover, $(G^*)_\Delta/\operatorname{Hom}(F_\Delta, X_\Delta)$ is divisible so that $(G^*)_\Delta = \operatorname{Hom}(F_\Delta, X_\Delta) + A((G_2)_P)_\Delta$. For future reference, we record this fact in the next lemma.

LEMMA 2.7. *Let G be a PX-group. Then there is a full homogeneous completely decomposable subgroup C of $(G^*)_\Delta$ of type equal to* type(X_Δ). *Moreover, $(G^*)_\Delta \doteq C + A((G_2)_P)_\Delta$.*

As a result of Proposition 2.6, we may define (the quasi-equality class) $G^* = \operatorname{Hom}(G_1, X) + A((G_2)_P) \subset \operatorname{Hom}(G, Q)$, where $G = G_1 + G_2$ is any PX-decomposition of the PX-group G. The definition highlights the role of the fixed set of primes P. If S is a set of primes such that G_S is free, then $(G^*)_{S \cap P}$ is divisible, while $(G^*)_{S \setminus P}$ is free. Similarly, if G_S is divisible, then $(G^*)_{S \setminus P}$ is divisible, while $(G^*)_{S \cap P}$ is free.

The next proposition is an amplification of Lemma 2.7.

PROPOSITION 2.8. *Let G be a PX-group. Then $(G^*)_P$ contains a full homogeneous completely decomposable subgroup C of type equal to type (X_P) such that $C + A((G_2)_P)_P = (G^*)_P$. In particular, $(G^*)_P/C$ is quasi-equal to a divisible torsion group.*

Proof. Note first that $A((G_2)_P)_P = A(G_2)$ is p-divisible for all primes

$p \in \Pi \setminus P$, since G_2 is reduced on $\Pi \setminus P$. Partition P as $P = \Delta \cup (P \cap D) \cup P \setminus I$. We will show the proposition holds when localized at each of the three sets in the partition. The result is true for the localization at Δ by Lemma 2.7. By definition, $(G_2)_{P \cap D}$ is divisible and $(G_1)_{P \cap D} = F_{P \cap D}$. Therefore, $A((G_2)_P)_{P \cap D}$ is free, while $\mathrm{Hom}(G_1, X)_{P \cap D}$ is homogeneous completely decomposable of type equal to $\mathrm{type}(X_{P \cap D})$. This gives the proposition on $P \cap D$. Finally, $(G_2)_{P \setminus I} = F_{P \setminus I}$, so that $A((G_2)_P)_{P \setminus I}$ is divisible, and the result is true on $P \setminus I$ as well.

We intend to show that the map $G \to G^*$ induces a contravariant functor on the category of PX-groups and quasi-homomorphisms. We begin with the following technical lemma.

LEMMA 2.9. *Let G be a PX-group with PX-dual G^* and denote $I^* = I(G^*)$, $D^* = D(G^*)$, and $\Delta^* = I^* \setminus D^*$. Then*

(a) $P \setminus I^* = D \cap P$ *and* $P \setminus I = D^* \cap P$,

(b) $\Delta = \Delta^*$,

(c) $I \setminus P = D \setminus P = D^* \setminus P = I^* \setminus P$.

Proof. (a) By definition, $(G_1)_p$ and X_p are Z_p-free for all $p \in P$. In this case, $\mathrm{Hom}(G_1, X)_p$ is Z_p-free also. Therefore, $p \in I^* \cap P$ if and only if $p \in I(A((G_2)_P))$. Thus, from the definition of Arnold duality, $p \in I^* \cap P$ if and only if $p \in P \setminus D$. Hence, $P \setminus I^* = P \cap D$. Similarly, $p \in P \cap D^*$ if and only if $p \in P \setminus I$. (b) From the definitions, $p \in \Delta$ if and only if $pG \neq G$ while $(G/F)_p$ is infinite. More specifically, $pG_2 \neq G_2$ while $(G_2/F)_p$ is infinite. Since these two properties are inherited by $A((G_2)_P)$, the result follows. (c) Note that $p \in D \setminus P$ if and only if $pX = X$ if and only if $p\,\mathrm{Hom}(G_1, X) = \mathrm{Hom}(G_1, X)$. Moreover, $p \in D^* \setminus P$ implies that $pG^* = G^*$, while $A((G_2)_P)$ is p-reduced. Therefore, $p\,\mathrm{Hom}(G_1, X) = \mathrm{Hom}(G_1, X)$ if and only if $p \in D^* \setminus P$. We have shown $D \setminus P = D^* \setminus P$. Since G is a PX-group, $I \setminus D \subset P$, which implies $I \setminus P = D \setminus P$. However, by part (b), $I \setminus D = I^* \setminus D^*$. Therefore, $I^* \setminus D^* \subset P$ and $I^* \setminus P = D^* \setminus P$.

PROPOSITION 2.10. *Let G be a PX-group. Then G^* is a PX-group.*

Proof. We must verify conditions (a)–(c) of Definition 2.1. Conditions (b) and (c) follow immediately from Lemma 2.9(c) and (b). For condition (a), observe that $FOT(G^*) = FOT(\mathrm{Hom}(G_1, X)) \leqslant \mathrm{type}(X)$.

As a first step in showing $*$ is functorial, we prove:

LEMMA 2.11. *Let G and H be PX-groups with PX-decompositions $G = G_1 + G_2$ and $H = H_1 + H_2$ and suppose $f\colon G \to H$. Then*

(a) $f(G_2) \doteq H_2$ *and*

(b) $f(G_1)_S \doteq (H_1)_S$, *where* $S = \Pi \backslash P \cup (D(G) \cap P) \cup P \backslash I(H)$.

Proof. (a) By the definition of G_2 (2.4), any torsion image of G_2 is quasi-equal to a group $\bigoplus_{p \in P} D_p$, where each D_p is a divisible p-group. The map f induces a map $G_2 \to H/H_2$. Moreover, the torsion group H/H_2 is quasi-equal to a homomorphic image of H_1. However, by the definition of H_1 (2.4), any torsion image of H_1 is P-reduced. Thus, $f(G_2) \doteq H_2$.

(b) By Definition 2.4, $(H_1)_T = H_T$, if $T = \Pi \backslash P \cup P \backslash I(H)$. Thus, $f(G_1)_T \doteq (H_1)_T$. On the other hand, if $D = D(G)$, $(G_1)_{D \cap P} = F_{D \cap P}$, so that $f(G_1)_{D \cap P} \doteq (H_1)_{D \cap P}$ as well.

The next result will allow us to utilize Proposition 2.8 to show that $*$ is an exact functor.

LEMMA 2.12. *Suppose* $0 \to U \to^{\alpha} V \to^{\beta} W \to 0$ *is an exact sequence of* Q-*vector spaces. Let* $U_i \subset U$, $V_i \subset V$, $W_i \subset W$, $i = 1, 2$, *be subgroups such that the induced sequence* $0 \to U_2 \to V_2 \to W_2 \to 0$ *is exact, and denote* $Y_0 = Y_1 + Y_2$ *for* $Y = U, V, W$. *Then the induced sequence* (E_1): $0 \to U_0 \to V_0 \to W_0 \to 0$ *is exact if and only if the induced sequence* (E_2): $0 \to U_0/U_2 \to V_0/V_2 \to W_0/W_2 \to 0$ *is exact.*

Proof. Without loss of generality, we may assume α is containment of U in V. The only if part is routine.

Suppose (E_2) is exact. To show the exactness of (E_1), the non-trivial part of the job is to show exactness at V_0. Suppose $x \in V_0$ satisfies $\beta(x) = 0$. By the exactness of (E_2), there exists $y \in U_0$ such that $x - y \in V_2$. Clearly, $\beta(x - y) = 0$, so $x - y \in U$ as well. Thus, $x - y \in U \cap V_2 = U_2$, and $x = (x - y) + y \in U_0$.

We can now prove the main result of this section.

THEOREM 2.13. *The map* $G \to G^*, f \to f^* = \mathrm{Hom}(f, Q)$ *provides an exact contravariant functor from the category of* PX-*groups and quasi-homomorphisms into itself.*

Proof. To show $*$ is a contravariant functor on the category of PX-groups and quasi-homomorphisms, let G and H be PX-groups and $f: G \to H$ a quasi-homomorphism. By Lemma 2.6, G^* and H^* are uniquely defined PX-groups, up to quasi-equality. We need to show that $f^*(H^*) \doteq G^*$. Suppose $G = G_1 + G_2$ and $H = H_1 + H_2$ are PX-decompositions of G and H, respectively. By Lemma 2.11(a), $f(G_2) \doteq H_2$, so by Arnold duality, $f^*(A(H_2)_P) \doteq A((G_2)_P) \subset G^*$. Similarly, by (part of) Lemma 2.11(b), $f(G_1)_S \doteq (H_1)_S$, where $S = \Pi \backslash P$, so by Warfield duality, $f^*W(H_1)_S \doteq W(G_1)_S$. It follows that $f^*(H^*)_S \doteq (G^*)_S$.

It remains to show $f^*(H^*)_P \doteq (G^*)_P$, for which we employ Proposition 2.8 and Lemma 2.12. Write $(H^*)_P = C + A(H_2)$, where C is full homogeneous completely decomposable subgroup of type equal to type(X_P). As we have already noted, $f^*(A(H_2)) \doteq A(G_2) \subset (G^*)_P$ by Arnold duality. However, $f^*(C) \doteq (G^*)_P$ is immediate from the fact that, by Proposition 2.8, $(G^*)_P$ also contains a full homogeneous completely decomposable subgroup of type equal to type(X_P).

For exactness, let $0 \to G \to H \to K \to 0$ be an exact sequence of PX-groups. It is routine that $K^* \to H^*$ is a monomorphism and that the composition $K^* \to H^* \to G^*$ is zero. To see that $H^* \to G^*$ is an epimorphism, first note that $A((H_2)_P) \to A((G_2)_P)$ is epic by Arnold duality and Lemma 2.11(a). Further, if $S = \Pi \setminus P$, then $W(H_1)_S \to W(G_1)_S$ is epic by Warfield duality and Lemma 2.11(b). Finally, by Proposition 2.8, $(H^*)_P = C_H + A(H_2)$ and $(G^*)_P = C_G + A(G_2)$ for full homogeneous completely decomposable subgroups C_H and C_G of type equal to type(X_P). Then $C_H \to C_G$ is a quasi-epimorphism by the properties of such groups so that $(H^*)_P \to (G^*)_P$ is a quasi-epimorphism as well.

To complete the proof of exactness at H^*, we first argue, as above, that $A((K_2)_P) \to A((H_2)_P) \to A((G_2)_P)$ and $W(K_1)_{\Pi \setminus P} \to W(H_1)_{\Pi \setminus P} \to W(G_1)_{\Pi \setminus P}$ are exact by the properties of Arnold and Warfield duality. As a consequence, we need only show the exactness of (E_1): $(K^*)_P \to^{\alpha} (H^*)_P \to^{\beta} (G^*)_P$. We have already noted that the composition $\beta\alpha = 0$. By Proposition 2.8, there is an exact sequence $0 \to C_K \to C_H \to C_G \to 0$ with C_Y a full subgroup of $(Y^*)_P$ and C_Y homogeneous completely decomposable of type equal to type(X_P) for $Y = G, H, K$. Moreover, the induced sequence (E_2): $0 \to (K^*)_P/C_K \to (H^*)_P/C_H \to (G^*)_P/C_G \to 0$ is an exact sequence of divisible torsion groups. Exactness of (E_1) then follows from Lemma 2.12.

3. The Dualizable PX-Groups

We will call a PX-group G *dualizable* provided G_P contains a full homogeneous completely decomposable subgroup C of type equal to type(X_P) such that G_P/C is divisible. It is easy to check that, like PX-groups, the dualizable PX-groups are closed under quasi-isomorphism, sums, summands, and homomorphic images.

By Proposition 2.8, if G is a PX-group, then G^* is a dualizable PX-group. Thus, if we wish $*$ to provide a duality, we must restrict ourselves to dualizable PX-groups.

LEMMA 3.1. *Let G be a dualizable PX-group. Then there is a full* *subgroup F of G such that, for each $p \in P$, $(G/F)_p \simeq (X/Z)_p^r \oplus Z(p^\infty)^{n-r}$, where $r = r_p = p-\operatorname{rank}(G)$.*

Proof. By definition, G_P contains a full homogeneous completely decomposable subgroup C of type equal to type (X_P) such that G_P/C is divisible. Without loss of generality, assume $C = X_p F$ for some full free subgroup F of G. Then for $p \in P$, $(G/F)_p = (C/F)_p + \operatorname{div}(G/F)_p$. Thus, $(G/F)_p = T \oplus \operatorname{div}(G/F)_p$, where T is a summand of $(C/F)_p$. The result follows.

PROPOSITION 3.2. *Let G be a dualizable PX-group. Then*

(a) $G^* = \operatorname{Hom}(G_1, X) + A((G_2)_P)$ *is a PX-decomposition of G^**

(b) $(G^*)^* \doteq G$ *under the natural evaluation map* $\operatorname{Hom}(\operatorname{Hom}(QG, Q), Q) \simeq QG$.

Proof. (a) Let $G_1^* = \operatorname{Hom}(G_1, X)$, $G_2^* = A((G_2)_P)$, and let $F^* = \operatorname{Hom}(F, A)$, a full free subgroup of $G_1^* \cap G_2^*$. By definition, $G^* \doteq G_1^* + G_2^*$. To show this is a PX-decomposition, we verify the two parts of Definition 2.4. Since X is P-reduced, so is G_1^*. Therefore, $\operatorname{div}[(G^*/F^*)_P] \doteq (G_2^*/F^*)_P \doteq (G_2^*/F^*)$, noting that G_2^* is quotient divisible and p-locally free for $p \notin P$. This establishes one part of 2.4.

To establish the other, note that since $(G_2^*/F^*)_{\Pi \setminus P} \doteq 0$, $(G^*/F^*)_{\Pi \setminus P} \doteq (G_1^*/F^*)_{\Pi \setminus P}$. It remains to show that $(G_1^*/F^*)_P$ is the reduced part of $(G^*/F^*)_P$. Since G is a dualizable PX-group, by Lemma 3.1, $(G/F)_P \doteq R \oplus M$, where for each $p \in P$, $R_p \cong (X/Z)_p^r$ and $M_p \simeq Z(p^\infty)^{n-r}$, with $n = \operatorname{rank}(G)$ and $r = r_p = p\text{-rank}(G)$. Without loss of generality, we will assume $(G/F)_P = R \oplus M$.

Fix $p \in P$. By definition, $(G_1/F)_p = R_p \simeq (X/Z)_p^r$ and $(G_2/F)_p = D_p = Z(p^\infty)^{n-r}$. It follows directly that $(G_1^*/F^*)_p = \operatorname{Hom}(G_1, X)_p/\operatorname{Hom}(F, Z)_p \simeq \operatorname{Hom}((G_1)_p, X_p)/\operatorname{Hom}(F_p, Z_p) \simeq (X/Z)_p^{n-r}$, while $(G_2^*/F^*)_p = (A(G_2)/F^*)_p \simeq Z(p^\infty)^r$ by the p-rank property of Arnold duality. However, $(G^*/F^*)_p = (G_1^*/F^*)_p + (G_2^*/F^*)_p \simeq (X/Z)_p^{n-r} \oplus Z(p^\infty)^r$ by Lemma 3.1. It follows that $(G^*/F^*)_p = (G_1^*/F^*)_p \oplus (G_2^*/F^*)_p$, and therefore that $(G_1^*/F^*)_P$ is the reduced part of $(G^*/F^*)_P$.

Statement (b) is a direct consequence of (a), the definition of $*$, and Arnold and Warfield duality.

THEOREM 3.3. *The functor $*$ is an exact duality on the category of dualizable PX-groups and quasi-homomorphisms.*

Proof. Apply Theorem 2.13 and Proposition 3.2(b).

THEOREM 3.4. *Let G be a PX-group and let e*: $QG \to \operatorname{Hom}(\operatorname{Hom}(QG, Q), Q)$

be the canonical evaluation homomorphism. Then $e(G) \dot{\subset} (G^*)^*$ *and* $e(G) \doteq (G^*)^*$ *if and only if G is a dualizable PX-group.*

Proof. Let F be a full free subgroup of G and define a group H with $G \subset H \subset QG$ by $H_{\Pi \setminus P} = G_{\Pi \setminus P}$ and $H_P = G_P + X_P F$. Then a direct calculation shows that H is a dualizable PX-group such that $H^* = G^*$. It follows that $e(G) \subset e(H) \doteq (H^*)^* = (G^*)^*$. Then $e(G) \doteq (G^*)^*$ if and only if $e(G) \doteq e(H)$; that is, $G \doteq H$ is a dualizable PX-group.

The final result of this section records the fact that we have generalized three distinct dualities. The careful reader will protest that Warfield duality is an exact duality in the category of X-locally free groups and homomorphisms, while our results are all in the quasi-category. Additional structure could be imposed to cover this problem, such as working with pairs (G, F) where F is a full free subgroup of the group G. Alternately, we could switch to modules over Dedekind domains, where Warfield duality requires the quasi-category [La2, Re]. We have eschewed these options in the interest of simplicity.

PROPOSITION 3.5. *Let G be a group.*

(a) *If G is quotient divisible, then G is a dualizable PX-group and* $G^* = A(G)$, *the Arnold dual, if we take* $P = \Pi$ *and* $X = Z$. *In this case,* $G_1 = F$ *and* $G_2 = G$.

(b) *If G is X-locally free, then G is a dualizable PX-group and* $G^* = \mathrm{Hom}(G, X)$, *the Warfield dual, if we take P to be empty and* $X = X$. *In this case,* $G_1 = G$ *and* $G_2 = F$.

(c) *If G is a* $(\sigma - \rho)$*-group, in the sense of Fomin* [Fo2], *then G is a dualizable PX-group and* G^* *is the Fomin dual of G if we take* $P = \Delta(G)$ *and let* X, X_1, *and* X_2 *be subgroups of Q with* $\mathrm{type}(X) = \mathrm{type}(X_1) + \mathrm{type}(X_2)$, *where* $\mathrm{type}(X_1) = \sigma$ *and* $\mathrm{type}(X_2) = FOT(G/F)$. *In this case, if* $G/F = D \oplus R$, *then* $G_1/F = D_{\Pi \setminus \Delta} \oplus R$ *and* $G_2/F = D_\Delta$.

Parts (a) and (b) are clear. Part (c) is a straightforward translation of Fomin's matrix equations into our context. This requires a more lengthy discussion than seems appropriate here. The interested reader is referred to [Fo2]. We present the definition of $(\sigma - \rho)$-group to illustrate the extent to which our results generalize those of Fomin. Let $\sigma \leqslant \rho$ be types with characteristics $(s_p) \in \sigma$, $(r_p) \in \rho$ chosen so that $s_p \leqslant r_p$ for all primes p. Fomin [Fo2] defines a group G to be a $(\sigma - \rho)$*-group* if there exists a full free subgroup F of G such that, for each prime p, $(G/F)_p \cong Z(p^{s_p})^k \oplus Z(p^{r_p})^{n-k}$, where $0 \leqslant k = k(p) \leqslant n = \mathrm{rank}\ G$. If P and X are as in (c) of the proposition, it is easy to check that G is a dualizable PX-group. Note that our definition of dualizable PX-group only requires a Fomin-type condition on $(G/F)_p$ to hold for primes $p \in \Delta$, with $s_p = \text{finite}$ and $r_p = \infty$ (Lemma 3.1).

As a final remark, we note that if G is a PX-group which is a Butler group, then $G^* = \text{Hom}(G_1, X) + A((G_2)_P)$ is also a Butler group since both $\text{Hom}(G_1, X)$ and $A((G_2)_P)$ are Butler (see Example 2.5). We assert without proof that, in this case, $*$ is an example of the well-known dualities on the quasi-homomorphism category of Butler groups with typeset contained in a fixed finite lattice T of types (see [Ri2]).

4. An Application

In [Fo1], Fomin studied those groups G such that every subgroup of infinite index in G is free, in particular every proper pure subgroup is free. We consider a generalization of this class. Let $\tau < \sigma$ be types and let t and s be positive integers. Define $H(\tau, \sigma, t, s)$ to be the class of all groups G such that G is a strongly indecomposable group of rank $t+s$, every proper pure subgroup of G of rank less than or equal to t is homogeneous completely decomposable of type τ, and every non-trivial torsion-free image of G of rank less than or equal to s is homogeneous completely decomposable of type σ.

It is an easy exercise to prove that a strongly indecomposable group G of rank $t+s$ is an element of $H(\tau, \sigma, t, s)$ if and only if every pure subgroup of rank exactly t and every torsion free factor of rank exactly s are homogeneous completely decomposable of types τ and σ, respectively.

To employ the duality of Section 3, we need to associate a set of primes P and a subgroup X of Q to each pair of types $\tau < \sigma$. Let h and k be height vectors belonging to the types τ and σ, with $h < k$. Let l be a height vector defined by $l(p) = h(p)$ if $k(p) = \infty$ and $l(p) = h(p) + k(p)$ if $k(p)$ is finite. Set $P(\tau, \sigma) = \{p \mid h(p) < k(p) = \infty\}$ and let $X(\tau, \sigma)$ be a subgroup of Q such that type $X(\tau, \sigma) = [l]$.

If $G \in H(\tau, \sigma, t, s)$, it is simple to verify that G is a dualizable PX-group, where $P = P(\tau, \sigma)$ and $X = X(\tau, \sigma)$.

Theorem 4.1. *Let $\tau < \sigma$ be types and let t and s be positive integers. Let $P = P(\tau, \sigma)$, $X = X(\tau, \sigma)$, and let $*$ be the PX dual. Regard $H(\tau, \sigma, t, s)$ and $H(\tau, \sigma, s, t)$ as full subcategories of the category of dualizable PX-groups and quasi-homomorphisms. Then $H(\tau, \sigma, t, s)$ and $H(\tau, \sigma, s, t)$ are dual via $*$.*

Proof. Let $G \in H(\tau, \sigma, t, s)$ and suppose $(E)\colon 0 \to K \to^f G^* \to^g L \to 0$ is an exact sequence of groups, with rank $K = s$, rank $L = t$. Let (QE) be the exact sequence of Q-spaces obtained by tensoring (E) with Q. We must show that K and L are homogeneous completely decomposable of types τ

and σ, repectively. By the duality of finite dimensional Q-spaces there exist subspaces V and W of QG, of dimensions t and s, respectively, and mappings v and w, such that $\mathrm{Hom}(-, Q)$ applied to $0 \to V \to^{v} QG \to^{w} W \to 0$ yields (QE). Let $A = v^{-1}(G)$, $B = w(G)$. Then (E'): $0 \to A \to^{v} G \to^{w} B \to 0$ is exact. Moreover, since $G \in H(\tau, \sigma, t, s)$, A and B are homogeneous completely decomposable of types τ and σ, respectively. In particular, A and B are PX-groups and we may apply $*$ to (E'). By Theorem 2.13 and the definition of (E'), $(E')^* = (E)$. Furthermore, using the definitions of P, X, and $*$, a direct computation shows that $L \simeq A^*$ and $K \simeq B^*$ are homogeneous completely decomposable of types σ and τ, respectively. This completes the proof.

Let σ be a type. A group G is called a *hyper-σ group* if every proper torsion-free image of G is homogeneous completely decomposable of type σ. Several characterizations of and results concerning the class of hyper-σ groups are presented in [GW]. Theorem 4.1 allows those results to be dualized to an appropriate class of groups. First we need the following simple lemma which shows that the homogeneous hyper-σ groups belong to one of the classes under consideration.

LEMMA 4.2. *Let G be a rank-n homogeneous group of type τ. Then G is a hyper-σ group if and only if $G \in H(\tau, \sigma, 1, n-1)$.*

Proof. Immediate from the definitions.

As a consequence of Lemma 4.2 and Theorem 4.1, if G is a hyper-σ group which is additionally homogeneous of type τ, then $G^* \in H(\tau, \sigma, n-1, 1)$, where $n = \mathrm{rank}\, G$. All of the results in [GW] on homogeneous hyper-σ groups can now be dualized to the class $H(\tau, \sigma, n-1), 1)$, the strongly indecomposable rank-n groups which are cohomogeneous of type σ (every rank one factor has type σ) and have every proper pure subgroup homogeneous completely decomposable of type τ. For example, consider the following result:

THEOREM 3.1 FROM [GW]. *Let G be a homogeneous group of type τ. Then G is a hyper-σ group if and only if G/A is homogeneous completely decomposable of type σ for some rank one pure subgroup A of G.*

Dualization gives us:

COROLLARY 4.3. *Let $\tau < \sigma$ be types and let G be a cohomogeneous group of type σ. Then every proper pure subgroup of G is homogeneous completely decomposable of type τ if and only if there exists some pure subgroup H of G such that* rank $H = n - 1$ *and H is homogeneous completely decomposable of type τ.*

Remark. Corollary 4.3 can be generalized. Suppose $\tau < \sigma$ are types and G is a group of rank n which is homogeneous of type τ and cohomogeneous of type σ. If t and s are positive integers with $t+s=n$, then $G \in H(\tau, \sigma, t, s)$ if and only if G contains a pure subgroup H of rank t such that H is cohomogeneous completely decomposable of type τ and G/H is homogeneous completely decomposable of type σ. A proof of this result does not seem relevant here.

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